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A local study of group classes [i]

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Abstract. It was established in [15] that the class of groups with a finite commutator subgroup can be locally described by locally graded groups having a bound on the length of particular chains of non-normal subgroups. This approximation was later extended to groups having a finite normal subgroup whose factor group has no non-permutable subgroups (see [18]).

The aim of this paper is to show that these approximating group classes behave better than the classes they approximate, and can be used to derive new results on these.

Keywords: hamiltonian group, quasihamiltonian group, group class defined by an iteration process

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Introduction

Let G be a group. A subgroup H of G is called *permutable* (in G) if $HK=KH$ for all subgroups K of G. Groups in which all subgroups are permutable, that is *quasihamiltonian groups*, have been completely characterized (see [36]), thus the problem arose of understanding the structure of groups for which the set of non-permutable subgroups is small in some sense (see for instance [9],[10],[13],[18],[25]).

In particular, in [18] the authors provided a further contribution to this topic by looking at quasihamiltonian groups in the general framework of group classes that can be obtained by iterating a restriction on non-permutable subgroups.

Let $\mathfrak{X}$ be a class of groups. Put $\overline{\mathfrak{X}}_1 = \mathfrak{X}$, and suppose by induction that a group class $\overline{\mathfrak{X}}_k$ has been defined for some positive integer k; then we denote by

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$\overline{\mathfrak{X}}_{k+1}$ the class consisting of all groups in which every non-permutable subgroup belongs to $\overline{\mathfrak{X}}_k$. Moreover, we put

$$\overline{\mathfrak{X}}_\infty = \bigcup_{k\geq 1} \overline{\mathfrak{X}}_k.$$

A similar construction was carried out in [15], where normality replaced permutability; the corresponding classes will be here denoted by $\mathfrak{X}_k$ for all k in $\mathbb{N}\cup\{\infty\}$.

This iteration has been particularly fruitful when applied to the class $\mathfrak{A}$ of all abelian groups. Indeed, it was for instance proved in [15] that the class of all locally graded $\mathfrak{A}_\infty$-groups completely describes the class of all groups with a finite derived subgroup, i.e. *finite-by-abelian groups*, while in [13] the authors proved that a group is *finite-by-quasihamiltonian*, that is, it has a finite normal subgroup with quasihamiltonian quotient, if and only if it is a locally graded $\overline{\mathfrak{A}}_\infty$-group; recall that a *locally graded group* is a group in which all non-trivial finitely generated subgroups contain proper subgroups of finite index. In what follows we usually refers to groups in $\mathfrak{A}_k$ as *k-hamiltonian groups* and to groups in $\overline{\mathfrak{A}}_k$ as *k-quasihamiltonian groups*; notice that 2-hamiltonian groups are the usual *metahamiltonian groups*, that is, groups in which every subgroup is either normal or abelian, while 2-quasihamiltonian groups where first studied in [9] under the name of *metaquasihamiltonian groups*.

This approach makes therefore possible to study the classes of finite-by-abelian groups and finite-by-quasihamiltonian groups in a local way; e.g. it was proved in [16] that locally graded groups with finitely many normalizers of non-$\mathfrak{A}_k$ subgroups are finite-by-abelian. The aim of this work is to give a further contribution to the aforementioned local study by looking at restrictions on the number of normalizers, on the number of conjugacy classes and on some kinds of infinite subgroups; this local study will also allow us to prove a number of new properties of the class of finite-by-quasihamiltonian groups.

Most of our notation is standard and can be found in [35]. For a full account of permutable subgroups and quasihamiltonian groups we refer to [36].

1 Restrictions on the normalizers

Let χ be any subgroup theoretical property. We say that χ is a *normal subgroup theoretical property* if

1. any normal subgroup of a group has the property χ in that group, and
2. any subgroup H of a group G having the property χ in G has the same property in any subgroup K of G containing it.

Easy examples of normal subgroup theoretical properties are "normality" and "permutability".

Now let $\mathfrak{X}$ be a class of groups. Put $\mathfrak{X}_{1,\chi} = \mathfrak{X}$, and suppose by induction that a group class $\mathfrak{X}_{k,\chi}$ has been defined for some positive integer k; then we denote by $\mathfrak{X}_{k+1,\chi}$ the class consisting of all groups in which every non-χ subgroup belongs to $\mathfrak{X}_{k,\chi}$; moreover, put

$$\mathfrak{X}_{\infty,\chi} = \bigcup_{k\geq 1} \mathfrak{X}_k.$$

It is obvious from the definitions that if χ is normal and $\mathfrak{X}$ is subgroup closed, then all group classes $\mathfrak{X}_{j,\,\chi}$ are subgroup closed and $\mathfrak{X}_{i,\chi} \subseteq \mathfrak{X}_{i+1,\chi}$ for all $i \geq 1$. We explicitly observe that in general the class of finite groups is contained in $\mathfrak{X}_{\infty,\chi}$, and that, if χ is normal and $\mathfrak{X}$ is subgroup closed, $\mathfrak{X}_{k+1,\chi}$ contains all groups with at most one normalizer of non-$\mathfrak{X}_{k,\chi}$ subgroups; in particular, it contains all *Dedekind groups* (that is, groups in which all subgroups are normal) and all groups whose subgroups have the $\mathfrak{X}_{k,\chi}$-property. The following theorem show that this last remark can be slightly extended.

Theorem 1. *Let χ be a normal subgroup theoretical property, let $\mathfrak{X}$ be any subgroup closed group class and let k be a positive integer. If G is any group having only finitely many normalizers of subgroups that are not in $\mathfrak{X}_{k,\chi}$, then G belongs to $\mathfrak{X}_{\infty,\chi}$.*

Proof. Of course, it can be assumed that G is infinite and it is not an $\mathfrak{X}_{k+1,\chi}$-group, so that it contains a subgroup which neither is an $\mathfrak{X}_{k,\chi}$-group nor has the property χ in G. As χ is a normal subgroup theoretical property, it follows that the normalizers

$$N_G(X_1), \dots, N_G(X_t)$$

of all subgroups of G which are neither $\mathfrak{X}_{k,\chi}$-groups nor have the χ-property in G must be proper subgroups of G. Therefore induction on the number of normalizers of non-$\mathfrak{X}_{k,\chi}$ subgroups show that for all $i = 1, \dots, t$ the subgroup $N_G(X_i)$ is an $\mathfrak{X}_{h_i,\chi}$-group for some positive integer h_i. Put

$$h = \max\{k, h_1, \dots, h_t\},$$

and let X be any subgroup of G which neither is an $\mathfrak{X}_{k,\chi}$-group nor has the property χ in G. Then

$$X \leq N_G(X) = N_G(X_i)$$

for some $i \leq t$, and so X is an $\mathfrak{X}_{h_i,\chi}$-group. Therefore all non-χ subgroups of G have the $\mathfrak{X}_{h,\chi}$-property, and hence G is an $\mathfrak{X}_{h+1,\chi}$-group. The statement is proved. $\boxed{QED}$

A direct application of the above result to the class of metaquasihamiltonian groups yields the following corollary.

Corollary 1. *Let G be a group with only finitely many normalizers of subgroups that are not metaquasihamiltonian. Then G is k-quasihamiltonian for some positive integer k.*

As a further consequence of Theorem 1, we may improve the above quoted result from [18], by showing that the same conclusion holds also in the case of locally graded groups which have only finitely many normalizers of subgroups that are not k-quasihamiltonian; this also generalizes the corresponding result of [16].

Corollary 2. *Let k be a positive integer and let G be a locally graded group which has only finitely many normalizers of subgroups that are not k-quasihamiltonian. Then there is a finite normal subgroup N such that G/N is quasihamiltonian.*

Proof. It follows from Theorem 1 that the group G is h-quasihamiltonian for some positive integer h. Then the main result of [18] implies the existence of a finite normal subgroup N such that G/N is quasihamiltonian. $\boxed{QED}$

The following alternative characterization of finite-by-quasihamiltonian groups follows from the above corollary and the main result of [18].

Theorem 2. *Let G be a locally graded group. Then G is finite-by-quasihamiltonian if and only if there is some positive integer k for which G has only finitely many normalizers of non k-quasihamiltonian subgroups.*

Finally, notice that the locally dihedral 2-group is a group having all its proper subgroups finite or abelian but not being finite-by-quasihamiltonian.

2 Restrictions on the conjugacy classes

For the sake of simplicity, we refer to the following properties for an arbitrary group class $\mathfrak{X}$ as follows:

(a) $\mathfrak{X}$ is a local group class;

(b) $\mathfrak{X}$ is subgroup closed;

(c) every $\mathfrak{X}$-group G contains normal subgroups $N \leq M$ such that N and G/M are Černikov while M/N is a soluble, hypercentral group;

(d) $\mathfrak{X}$ is an accessible group class.

Recall that a group class $\mathfrak{X}$ is *accessible* if every locally graded group whose proper subgroups belong to $\mathfrak{X}$ is either finite or an $\mathfrak{X}$-group, or equivalently if any locally graded minimal non-$\mathfrak{X}$ group is finite. In particular, a group class $\mathfrak{X}$ containing all finite groups is accessible if and only if there are no minimal non-$\mathfrak{X}$ groups in the universe of locally graded groups. It is easy to show that abelian groups form an accessible group class, and $\mathfrak{A}$ shares such a property with other relevant classes of groups. Further information on accessible group classes can be found in [28].

We say that a group class $\mathfrak{X}$ is *conjugacy well-behaving* if it satisfies properties (a)-(d). It follows from results in [15] and [13] that locally graded k-(quasi)hamiltonian groups make for conjugacy well-behaving group classes for all positive integers k (see [36]), but observe that the example of the locally dihedral 2-group show that the classes of finite-by-abelian and finite-by-quasi-hamiltonian groups are not accessible.

Lemma 1. *Let $\mathfrak{X}$ be a conjugacy well-behaving group class and let G be a locally graded, non-periodic, non-$\mathfrak{X}$ group. Then G admits an infinite descending chain of finitely generated, non-periodic, non-$\mathfrak{X}$ subgroups.*

Proof. As the class $\mathfrak{X}$ is local and subgroup closed, then G contains a finitely generated non-$\mathfrak{X}$ subgroup F. Adding an aperiodic element we can clearly suppose F being non-periodic. Assume by way of contradiction that every proper subgroup of finite index of F is an $\mathfrak{X}$-group and let E be such a subgroup. Then E is an $\mathfrak{X}$-group and so *polycyclic-by-finite*, that is, it contains a normal polycyclic subgroup of finite index. Therefore, F is polycyclic-by-finite and hence every subgroup of F is the intersection of subgroups of finite index. In particular, every proper subgroup of F is contained in a proper subgroup of finite index and it must have the $\mathfrak{X}$-property. As $\mathfrak{X}$ is an accessible group class, it follows that F is either finite or an $\mathfrak{X}$-group, a contradiction in both cases.

Thus, we have proved that any non-periodic finitely generated non-$\mathfrak{X}$ subgroup of G has a proper subgroup of finite index which is still not in $\mathfrak{X}$. This obviously concludes the proof of the statement. $\boxed{QED}$

Corollary 3. *Let $\mathfrak{X}$ be a conjugacy well-behaving group class and let G be an infinite locally graded non-$\mathfrak{X}$ group satisfying the minimal condition on non-$\mathfrak{X}$ subgroups. Then G is Černikov.*

Proof. It is possible to assume G being a minimal counterexample to the statement; in particular, G is neither an $\mathfrak{X}$- nor a Černikov group but all its proper subgroups are either $\mathfrak{X}$- or Černikov groups.

By Lemma 1, the group G is periodic and hence also locally finite, not being G finitely generated. Indeed, if it were finitely generated then it would have a proper subgroup of finite index which is either Černikov (not possible) or an $\mathfrak{X}$-group; in this latter case it would be also polycyclic-by-finite and hence even finite, still a contradiction.

Let N be any Černikov normal subgroup of G not having the $\mathfrak{X}$-property. Then all proper subgroups of G/N are Černikov and G/N satisfies the minimal condition on subgroups. As G is locally finite, it follows that G/N itself is Černikov and hence even G is such, a contradiction.

Thus we may assume all proper normal subgroups of G have the $\mathfrak{X}$-property. As the class of $\mathfrak{X}$-groups is local, it is easy to see that G contains a maximal proper normal subgroup M. Moreover, since every $\mathfrak{X}$-group has a normal soluble subgroup of finite index (i.e. it is *soluble-by-finite*) and has a normal hypercentral subgroup whose factor group is Černikov (recall that a periodic group of automorphisms of a Černikov group is Černikov), it follows from a combination of [30] and [34] that G/M must be finite. Therefore G is soluble-by-finite since M is an $\mathfrak{X}$-group too.

Let X be any finite subgroup of G which is not in $\mathfrak{X}$. An application of Lemma 4.5 of [6] yields that there exists an abelian subgroup A of G such that $A^X = A$ and A does not satisfy the minimal condition on subgroups. Then the socle S of A must be infinite and X-invariant. Let B be any proper subgroup of finite index in S having index strictly larger than $|X|$. Then the core B_X of B in SX is such that $B_X X$ is strictly contained in SX and hence in G. Thus $B_X X$ is not an $\mathfrak{X}$-group and it does not satisfy the minimal condition on subgroups. This contradiction proves the statement. $\boxed{QED}$

Corollary 4. *Let $\mathfrak{X}$ be a conjugacy well-behaving group class and let G be an infinite locally graded non-$\mathfrak{X}$ group. If G does not contain proper infinite non-$\mathfrak{X}$ subgroups, then it is Černikov.*

Let $\mathfrak{X}$ be a conjugacy well-behaving group class and let G be a group having at most one conjugacy class of non-$\mathfrak{X}$ subgroups. As $\mathfrak{X}$ is subgroup closed, then such a conjugacy class must be that of G and hence all proper subgroups of G have the $\mathfrak{X}$-property; being $\mathfrak{X}$ accessible, it follows that G either is finite or an $\mathfrak{X}$-group. The following theorem shows that the above remark can be slightly improved.

Theorem 3. *Let $\mathfrak{X}$ be a conjugacy well-behaving group class and let G be an infinite locally graded group having finitely many conjugacy classes of non-$\mathfrak{X}$ subgroups, then G is an $\mathfrak{X}$-group.*

Proof. By Proposition 3.3 of [19] and the fact that $\mathfrak{X}$-groups locally satisfies the maximal condition on subgroups, it follows that G itself locally satisfies the maximal condition on subgroups. Now Lemma 4.7 of [6] yields that G satisfies the minimal condition on non-$\mathfrak{X}$ subgroups and Corollary 3 let us assume that G is Černikov. By hypothesis there is an upper bound k for the orders of finite non-$\mathfrak{X}$ subgroups of G. If X is any finite subgroup of G, there is a finite subgroup Y of G such that $X \leq Y$ and $|Y| > k$; then Y has the $\mathfrak{X}$-property and the local closure of $\mathfrak{X}$-groups completes the proof of the statement. $\boxed{QED}$

Corollary 5. *Let k be a positive integer and let G be an infinite locally graded group with finitely many conjugacy classes of non k-(quasi)hamiltonian subgroups. Then G is k-(quasi)hamiltonian.*

As in the previous section we notice that the locally dihedral 2-group shows that the above corollary does not hold for both the classes $\mathfrak{A}_\infty$ and $\overline{\mathfrak{A}}_\infty$.

Local graduation in Corollaries 3 and 5 is essential. Indeed, if n is any positive integer and H_n is any finite $(n+1)$-hamiltonian group of odd order which is not n-quasihamiltonian (see next section for such examples), then Theorem 35.1 of [33] can be used to construct a (non locally graded) group G with the following properties:

1. H_n embeds in G;
2. all proper subgroups of G are finite and either cyclic or conjugated to a subgroup of the chosen embedding of H_n.

Clearly, G satisfies the minimal condition on subgroups and has finitely many conjugacy classes of non-abelian subgroups. However, G is $(n+2)$-hamiltonian but not $(n+1)$-quasihamiltonian.

Finally, since as we previously remarked the class of finite groups is contained in that of ∞-(quasi)hamiltonian groups, we obtain the following corollary (it is not clear if the local graduation here is essential or not).

Corollary 6. *Let k be a positive integer and let G be a locally graded group with finitely many conjugacy classes of non k-(quasi)hamiltonian subgroups. Then G is ∞-(quasi)hamiltonian.*

3 Restrictions on infinite subgroups

In this section we restrict our attention to infinite subgroups of a group in the spirit of [3] where it was proved that a non-abelian locally graded group in which all infinite subgroups are (either normal or) abelian is (either metahamiltonian or) Černikov. We now wish to generalize these results to k-(quasi)hamiltonian groups, and, in order to do so, we give the following definitions.

Let $\mathfrak{X}$ be a class of groups. We define a sequence of group classes

$$\overline{\mathfrak{X}}_1^{\infty}, \overline{\mathfrak{X}}_2^{\infty}, \ldots, \overline{\mathfrak{X}}_k^{\infty}, \ldots$$

by putting $\overline{\mathfrak{X}}_1^{\infty} = \mathfrak{X}$ and choosing $\overline{\mathfrak{X}}_{k+1}^{\infty}$ as the class of all groups whose infinite non-permutable subgroups belong to $\overline{\mathfrak{X}}_k^{\infty}$. The corresponding classes for normality will be denoted by $\mathfrak{X}_k^{\infty}$.

Now we add to the list of properties in the beginning of the previous section the following ones.

(e) $\mathfrak{X}$ is a closed by taking homomorphic images.

(f) Every locally graded $\mathfrak{X}$-group is soluble-by-finite.

(g) Every locally graded $\mathfrak{X}_\infty$-group satisfies (c).

(h) $\mathfrak{X}$ is a Robinson class.

(i) Every locally graded group in which all proper infinite subgroups have the $\mathfrak{X}$-property is either Černikov or an $\mathfrak{X}$-group.

Here by *Robinson class* it is meant a group class $\mathfrak{Y}$ such that every finitely generated hyper-(abelian or finite) group whose finite homomorphic images have the $\mathfrak{Y}$-property belongs to the class and contains a polycyclic subgroup of finite index (see [15] for further details but recall that a group is *hyper-(abelian or finite)* if it has an ascending normal series whose factors are either abelian or finite).

Theorem 4. *Let $\mathfrak{X}$ be a group class satisfying* (a),(b),(e)-(g). *Then for each positive integer k one has that*

1. *the class $\overline{\mathfrak{X}}_k^{\infty}$ satisfies* (b),(e)-(f), *and that*
2. *any locally graded $\overline{\mathfrak{X}}_k^{\infty}$-group is either a Černikov group or has the $\overline{\mathfrak{X}}_k$-property.*

Proof. We will work by induction on k, being the result true when $k = 1$. Let $k \geq 2$ and suppose the result true for positive integers strictly smaller than k.

It is obvious that the class $\overline{\mathfrak{X}}_k^{\infty}$ is closed by subgroups and homomorphic images. Now, let G be a locally graded $\overline{\mathfrak{X}}_k^{\infty}$-group which is not Černikov: we begin by showing that G contains a locally soluble normal subgroup which is not Černikov; to this aim it is clearly enough to show that G contains a soluble subgroup of finite index.

If G'' were not perfect, then G''' would be either an $\overline{\mathfrak{X}}^{\infty}_{k-1}$-group and hence G would be soluble-by-finite by induction, or G/G''' would be quasihamiltonian and hence metabelian, a contradiction. Thus we may assume $G'' = G'''$ is infinite.

Let N be any proper normal subgroup of G''. Then it must have the $\overline{\mathfrak{X}}^{\infty}_{k-1}$-property, otherwise G'' would not be perfect, and by induction N is therefore soluble-by-finite. Let S_N be the largest normal soluble subgroup of N. Then $C_{G''}(N/S_N)$ has finite index in G'' and, would it be proper, it would be soluble-by-finite, as well as G. Thus we may always assume

$$G'' = C_{G''}(N/S_N),$$

which means that N/S_N is abelian and so $N = S_N$ is soluble.

If G'' coincides with the product of all its proper normal subgroups, then G'' would be locally soluble. In this case, if it were also Černikov, G would turn out to be soluble-by-finite.

It remains to deal with the case in which the product S of all proper normal subgroups of G'' is still a proper subgroup; then G''/S is obviously a simple group, and, as such, it does not contain any proper non-trivial permutable subgroup. Being S soluble, it is possible to assume G''/S infinite and not finitely generated (see [32]). Since all proper subgroups of G''/S are by induction either Černikov or $\overline{\mathfrak{X}}^{\infty}_{k-1}$ it follows that G''/S is a locally (soluble-by-finite) group whose proper subgroups are soluble-by-finite. It follows from Theorem B of [12] that G''/S is periodic, but in this case all its proper subgroups have a normal hypercentral subgroup whose factor group is Černikov and the usual combination of [30] and [34] yields a contradiction. This contradiction shows that G contains a locally soluble normal subgroup which is not Černikov.

Let F be any finite subgroup of G not having the $\overline{\mathfrak{X}}_{k-1}$-property. An application of Lemma 4.5 of [6] yields that there exists an abelian subgroup A of G such that $A^F = A$ and A does not satisfy the minimal condition on subgroups.

Let g be any element of G. Suppose first that A contains an aperiodic element. Then it is easy to find a non-trivial polycyclic, torsion-free abelian F-invariant subgroup B. Thus for every positive integer n, the subgroup B^nF is not in $\overline{\mathfrak{X}}^{\infty}_{k-1}$, otherwise

$$B^nF/B^n \simeq F$$

would be an $\overline{\mathfrak{X}}_{k-1}$-group, and hence it is permutable in G.

If g has infinite order and $\langle g\rangle \cap BF = \{1\}$, then g normalizes B^nF for all $n \geq 1$ and hence it normalizes their intersection, namely, F. In any other case, B has finite index in $\langle g\rangle BF$ and hence it is possible to assume B is even $\langle g\rangle$-invariant. Let N be any normal subgroup of $\langle g\rangle BF$ having finite index m. Then B^m is contained in N and B^mF is permutable in G as well as FN/N is permutable in $\langle g\rangle BF/N$. Theorem A of [31] yields that F is permutable with $\langle g\rangle$.

Assume A is periodic. The socle S of A must be infinite and hence it contains an F-invariant subgroup H such that $H \cap F = \{1\}$. Again, if g is aperiodic, then it normalizes every infinite subgroup of FH containing F and hence F. Let g be periodic and notice that H has finite index in the subgroup $\langle g\rangle FH$. If $\langle g, F\rangle$ were infinite, then $H \cap \langle g, F\rangle$ would have finite index in $\langle g, F\rangle$ and hence it would be finite, a contradiction. Thus $\langle g, F\rangle$ is finite and we may find an infinite $\langle g, F\rangle$-invariant subgroup N of H such that $N \cap \langle g, F\rangle = \{1\}$. Now,

$$\langle g, F\rangle \simeq \langle g, F\rangle N/N$$

and hence F permutes with $\langle g\rangle$ since FN is permutable in G.

Therefore in any case F permutes with an arbitrary element g of G, so that F is permutable in G.

Take any proper infinite subgroup X of G which is not permutable in G. By induction on k, X is either an $\overline{\mathfrak{X}}_{k-1}$- or a Černikov group. Suppose it is Černikov and not in $\overline{\mathfrak{X}}_{k-1}$. Then X is the union of an ascending chain of finite non-$\overline{\mathfrak{X}}_{k-1}$ subgroups. By what we have just proved, however, each of these subgroups is permutable in G, making X also permutable in G. This contradiction shows that G has the $\overline{\mathfrak{X}}_k$-property.

Finally, since $\mathfrak{X}$ satisfies (a),(b),(e) and (f), it follows from Lemma 8 of [13] that $\overline{\mathfrak{X}}_k$ is made by soluble-by-finite groups and hence any $\overline{\mathfrak{X}}_k^\infty$-group, which is either Černikov or has the $\overline{\mathfrak{X}}_k$-property, turns out to be soluble-by-finite. $\boxed{QED}$

Theorem 5. *Let $\mathfrak{X}$ be group class satisfying* (a),(b),(e)-(g)*. Then for each positive integer k one has that*

1. *the class $\mathfrak{X}_k^\infty$ satisfies* (b),(e)-(f)*, and that*
2. *any locally graded $\mathfrak{X}_k^\infty$-group is either a Černikov group or has the $\mathfrak{X}_k$-property.*

Proof. We will work by induction on k, being the result true when $k = 1$. Let $k \geq 2$ and suppose the result true for positive integers strictly smaller than k.

Let G be any locally graded $\mathfrak{X}_k^\infty$-group. It follows from Theorem 4 that G is soluble-by-finite. Let S be a soluble normal subgroup of finite index and suppose G is not Černikov, so that S cannot be Černikov. If F is a finite subgroup of G which is not an $\mathfrak{X}_{k-1}$-group, an application of Lemma 4.5 of [6] yields that there exists an abelian subgroup A of G such that $A^F = A$ and A does not satisfy the minimal condition on subgroups.

Let a be an aperiodic element of A. Then the subgroup $F_1 = \langle a \rangle^F$ is finitely generated, and we can find a non-trivial characteristic torsion-free subgroup E of F_1. If n is any positive integer, then $E^n F$ does not have the $\mathfrak{X}_{k-1}^{\infty}$-property, since $E^n \cap F = \{1\}$ and

$$E^n F / E^n \simeq F \notin \mathfrak{X}_{k-1}.$$

Thus $E^n F$ is normal in G. Since E is free abelian, it follows that the intersection of the subgroups $E^n F$ for any positive integer n is still normal in G, but, it is clear that this intersection is just F.

Suppose therefore that A is periodic, so that the socle S of A is infinite. If M is any F-invariant subgroup of finite index of S with $F \cap M = \{1\}$, then, as before, FM does not have the $\mathfrak{X}_{k-1}^{\infty}$-property, and thence FM is normal in G. Since S is residually finite and every subgroup of finite index in S contains an F-invariant subgroup of finite index having trivial intersection with F, it obviously follows that F is normal in G also in this case.

Finally, let X be any infinite subgroup of G which is not an $\mathfrak{X}_{k-1}$-group. Then it must be Černikov by induction and, as $\mathfrak{X}_{k-1}$ is a local class (see [15]), it follows that X is a union of finite non-$\mathfrak{X}_{k-1}$ subgroups. However, we showed above that such subgroups must be normal and hence X is normal in G. This shows that G has the $\mathfrak{X}_k$-property. $\boxed{QED}$

Corollary 7. *Let $\mathfrak{X}$ be a group class satisfying* (a)-(h)*. Then the classes $\overline{\mathfrak{X}}_k^{\infty}$ and $\mathfrak{X}_k^{\infty}$ satisfy* (i) *for all positive integers k.*

Proof. Let k be any positive integer and let G be a locally graded group whose proper infinite subgroups are $\overline{\mathfrak{X}}_k^{\infty}$-groups. Then all proper subgroups are either $\overline{\mathfrak{X}}_k$- or Černikov groups by Theorem 4. In particular, G satisfies the minimal condition on non-$\overline{\mathfrak{X}}_k$ subgroups. It follows from results in [13] that the class $\overline{\mathfrak{X}}_k$ is conjugacy well-behaving and hence Corollary 3 yields that G is either Černikov or has the $\overline{\mathfrak{X}}_k^{\infty}$-property.

The above arguments works fine also for the class $\mathfrak{X}_k$ having care to replace Theorem 4 by Theorem 5 and [13] by [15]. $\boxed{QED}$

It follows from very well-known results (and from those of [3] and [15]) that the class $\mathfrak{A}$ of all abelian groups satisfies (a)-(h), so that we have the following corollaries.

Corollary 8. *Let G be a locally graded $\mathfrak{A}_k^{\infty}$-group. Then G is either Černikov or k-hamiltonian.*

Corollary 9. *If G is a locally graded group whose proper infinite subgroups are $\mathfrak{A}_k$-groups, then G is either Černikov or k-hamiltonian.*

Corollary 10. *Let G be a locally graded $\overline{\mathfrak{A}}_k^\infty$-group. Then G is either Černikov or k-quasihamiltonian.*

Corollary 11. *If G is a locally graded group whose proper infinite subgroups are $\overline{\mathfrak{A}}_k$-groups, then G is either Černikov or k-quasihamiltonian.*

Local graduation in the above corollaries cannot be removed, as we now show. Let n be any positive integer and choose an odd prime p_n. Then by Dirichlet's theorem there are infinitely many odd primes $\equiv 1 \bmod p_n$; choose n of them $\{q_1, \dots, q_n\}$ which are distinct. Let P_n be a cyclic group of order p_n, let D_n be an abelian group of order $q_1 \cdot \ldots \cdot q_n$ and let

$$G_n = P_n \ltimes D_n$$

be the natural semidirect product of those groups. It is easy to see that G_n is $(n+1)$-hamiltonian but not n-quasihamiltonian. At this point, with the help of Theorem 35.1 of [33], we can construct a (non locally graded) group G with the following properties:

1. G_m embeds in G for all m;
2. all proper subgroups of G are finite and either cyclic or isomorphic to a subgroup of one of the G_m.

Clearly, G belongs to $\mathfrak{A}_2^\infty$ but not to $\overline{\mathfrak{A}}_\infty$ and surely it is not a Černikov group.

The following proposition shows that we can say something more in the Černikov case whenever all infinite proper subgroups are k-(quasi)hamiltonian groups for a fixed integer k.

Proposition 1. *Let G be an infinite Černikov group. If all infinite proper subgroups of G are k-(quasi)hamiltonian for some fixed integer k but G is not k-(quasi)hamiltonian, then the following conditions hold:*

1. *the finite residual J has no infinite proper G-invariant subgroup and it is in particular a p-group for some prime p;*
2. *either all infinite proper subgroups of G are abelian and G/J is cyclic of prime power order, or G is a finite extension of a central subgroup of type p^∞.*

Proof. We will prove this result for the class of k-quasihamiltonian groups, the analogous result for k-hamiltonian groups being proved in an essentially identical way.

As G is not an $\overline{\mathfrak{A}}_k$-group, it contains a finite subgroup E not having the $\overline{\mathfrak{A}}_k$-property. Let J be the finite residual of G, and let K be any infinite G-invariant subgroup of J. Then EK is an infinite non-$\overline{\mathfrak{A}}_k$ subgroup of G and so $EK = G$. It follows that J/K is finite, and hence $J = K$. Therefore J has no infinite proper G-invariant subgroups.

Let X be any proper subgroup of G containing J; in particular, X is an $\overline{\mathfrak{A}}_k$-group. As such it contains a finite normal subgroup N with X/N quasihamiltonian. Clearly the p-component of X/N is abelian, being of infinite exponent, so that X has a finite commutator subgroup. Thus $J \leq Z(X)$ and X is abelian when X/J is cyclic.

Let Y be any infinite proper subgroup of G. If $YJ = G$, then the intersection $Y \cap J$ is an infinite normal subgroup of G, so that

$$Y \cap J = J \quad \text{and} \quad J \leq Y,$$

a contradiction. Thus YJ is properly contained in G, so that $J \leq Z(YJ)$ and hence all infinite proper subgroups are abelian whenever G/J is cyclic of prime power order.

If G/J is not cyclic of prime power order, then $\langle J, x\rangle$ is abelian for all $x \in G$ of prime power order and hence J is central in G. $\boxed{QED}$

4 Restrictions on uncountable subgroups

The imposition of a "good" property to the uncountable subgroups of a group has usually a strong impact on the group itself. This was first remarked in [21] and since then a lot have been discovered (see for instance [5, 23, 25, 27]). In particular, it was proved in [21] that the requirement for all proper uncountable subgroups of an uncountable group to be finite-by-abelian usually implies the group itself to be finite-by-abelian; an analogous result for metahamiltonian(-by-finite) groups was proved in [27]. In this section we give a further contribution to the topic but first we need to recall some terminology and derive some basic facts.

A group class $\mathfrak{X}$ is said to be *countably recognizable* if, whenever all countable subgroup of a group G belong to $\mathfrak{X}$, then G itself is an $\mathfrak{X}$-group. Countably recognizable classes of groups were introduced by Baer in [1] and were studied by many authors (see [17, 22, 26, 24] for an overview of the subject). Among the countably recognizable group classes there are of course the local classes, so that the classes of k-(quasi)hamiltonian groups are countably recognizable. It is easy to see that the class of groups with a finite commutator subgroup is countably recognizable and Corollary 3.4 of [22] yields that also the class of finite-by-abelian-by-finite groups is countably recognizable; here a *finite-by-abelian-by-finite group* is a group G having a subgroup of finite index H such that H' is finite. It follows from Lemma 2.1 of [22] and the results in [18] that the class of finite-by-quasihamiltonian groups is countably recognizable. Moreover, we observe that Theorem 3.2 of [22] and the results in [18] show that the class of finite-by-quasihamiltonian-by-finite groups is countably recognizable; here, by *finite-by-quasihamiltonian-by-finite* it is meant a group with a subgroup of finite index which is finite-by-quasihamiltonian; we state these two facts as a proposition.

Proposition 2. *Both the classes of finite-by-quasihamiltonian groups and finite-by-quasihamiltonian-by-finite groups are countably recognizable.*

Let $\aleph$ be an uncountable cardinal number and let $\mathfrak{Y}$ be a local group class inherited by homomorphic images and subgroups such that every group G in $\mathfrak{Y}$ has a quasicentral subgroup N such that $|G/N| < \aleph$; recall that a subgroup N of a group G is *quasicentral* in G if every subgroup of N is normal in G. We say that a group class $\mathfrak{X}$ is an $(\aleph, \mathfrak{Y})$ *well-behaving group class* whenever it satisfies the following properties:

1. it is countably recognizable group class;
2. every group G in $\mathfrak{X}$ having cardinality $\aleph$ contains a normal subgroup M of cardinality strictly smaller than $\aleph$ such that $G/M \in \mathfrak{Y}$.

Before stating the main result of this section, note that, due to the existence of *Jónsson groups*, i.e. groups of uncountable cardinality in which all proper subgroups have strictly smaller cardinality, it is necessary to require the absence of simple homomorphic images of cardinality $\aleph$: under such an hypothesis every group of uncountable cardinality $\aleph$ has a proper subgroup of cardinality $\aleph$ (see [21] for more details).

Theorem 6. *Let $\aleph$ be a cardinal with cofinality* $\mathrm{cf}(\aleph)$ *strictly larger that $\aleph_0$, let $\mathfrak{X}$ be an $(\aleph, \mathfrak{Y})$ well-behaving group class and let G be a group of cardinality $\aleph$ whose proper subgroups of cardinality $\aleph$ are $\mathfrak{X}$-groups. If G has no simple homomorphic images of cardinality $\aleph$, then G has the property $\mathfrak{X}$.*

Proof. Assume for a contradiction that the statement is false, so that G contains a countable subgroup E which is not an $\mathfrak{X}$-group. Suppose G has no proper normal subgroup of cardinality $\aleph$. Then the fact that G has no simple homomorphic image of cardinality $\aleph$ and that $\mathrm{cf}(\aleph) > \aleph_0$ yield that E is contained in a proper normal subgroup H of G. Then N has cardinality strictly smaller than $\aleph$ and so G/H contains a proper subgroup of cardinality $\aleph$, which means that E is contained in such a subgroup, which is a contradiction.

Thus G contains a proper normal subgroup N of cardinality $\aleph$ and clearly $G = NE$. Suppose G/N is not finitely generated. Being countable, G/N admits only countably many finitely generated subgroups and each of their numerator, say F, contains a normal subgroup N_F of cardinality smaller than $\aleph$ and such that F/N_F belongs to $\mathfrak{Y}$. Obviously the normal closure N_F^G of N_F in G has cardinality strictly smaller than $\aleph$.

If $\mathcal{F}$ is the set of all numerators of finitely generated subgroups in G/N, then the subgroup

$$M = \langle N_F^G \,:\, F \in \mathcal{F}\rangle$$

is a normal subgroup of cardinality strictly smaller than $\aleph$, as $\mathrm{cf}(\aleph) > \aleph_0$. In the factor group G/M, every finitely generated subgroup is now an $\mathfrak{Y}$-group. Thus G/M itself is a $\mathfrak{Y}$-group of cardinality $\aleph$ and in particular it contains a quasinormal subgroup H/M with $|G/H| < \aleph$. It is now easy to find in H/M a G-invariant subgroup L/M of cardinality $\aleph$ such that G/L has also cardinality $\aleph$. It follows that every countable subgroup of G is contained in a proper subgroup of cardinality $\aleph$ and hence that G has the $\mathfrak{X}$-property.

Now we may assume G/N is finitely generated but not cyclic. If x is any element of G, the subgroup $N_x = \langle x, N\rangle$ is properly contained in G, so that it has the $\mathfrak{X}$-property. Thus there are normal subgroups $K_x \leq H_x$ of N_x contained in N such that both N_x/H_x and K_x have cardinality strictly smaller than $\aleph$ and H_x/K_x is quasicentral in N_x/K_x. Let $\mathcal{E}$ be a system of generators of E and put

$$K = \langle K_x^G \,:\, x \in \mathcal{E}\rangle \quad \text{and} \quad H = \bigcap_{x\in\mathcal{E}} H_x.$$

Then it is clear that HK/K has cardinality $\aleph$ and it is quasicentral in G/K. It is then possible to find a subgroup L/K of HK/K such that

$$|L/K| = \aleph = |HK/L|.$$

But this is clearly a contradiction as LE is a proper subgroup of G of cardinality $\aleph$.

Therefore it is legit to assume G/N cyclic of prime power order. Since N is an $\mathfrak{X}$-group it contains normal subgroups $H \leq K$ such that $|H|, |N/K| < \aleph$ and K/H is quasicentral in N/H. By Lemma 2.8 of [27] it is easy to find a normal subgroup L of N such that $H \leq L$, $|N/L| < \aleph$ and N/L is not finitely generated. Thus even G/L_G has cardinality $< \aleph$ and is not finitely generated. However, we showed above that such a situation leads to a contradiction. The theorem is proved. $\boxed{QED}$

Corollary 12. *Let $\aleph$ be a cardinal with cofinality* $\mathrm{cf}(\aleph)$ *strictly larger that* $\aleph_0$ *and let G be a group of cardinality $\aleph$ whose proper subgroups of cardinality*

$\aleph$ are finite-by-(quasi)hamiltonian. If G has no simple homomorphic images of cardinality $\aleph$, then G is finite-by-(quasi)hamiltonian.

If the hypothesis of countable recognizability of $\mathfrak{X}$ is replaced by that of being a local class, then the hypothesis on the cofinality can be dropped out in the statement of Theorem 6: this follows using the very same arguments of the last part of the proof. Thus we get the following theorem.

Corollary 13. *Let k be a positive integer, $\aleph$ be an uncountable cardinal and let G be a group of cardinality $\aleph$ whose proper subgroups of cardinality $\aleph$ are locally graded k-(quasi)hamiltonian groups. If G has no simple homomorphic images of cardinality $\aleph$, then G is k-(quasi)hamiltonian.*

5 Restrictions on subgroups of infinite rank

A group is said to have *finite rank* if there is a finite uniform upper bound for the minimum number generators of finitely generated subgroups; if there is no such a bound, one says that the group has *infinite rank*. In this last section we tackle the problem of groups whose proper subgroups of infinite rank are k-(quasi)hamiltonian for a fixed positive integer k. As before, this study is inspired by many other similar ones (see for instance [7, 11, 13, 14]).

In order to state the main results of this section we need to straighten (c) to the following property concerning a group class $\mathfrak{X}$:

(c') every $\mathfrak{X}$-group G contains normal subgroups $L \leq N \leq M$ such that L and G/M are Černikov, N/L is quasicentral in G/L, M/L is hypercentral and M/N is soluble of finite rank.

We will also need the following definition. Let $\mathfrak{D}$ be the class of all periodic locally graded groups, and let $\overline{\mathfrak{D}}$ be the closure of $\mathfrak{D}$ by the operators $\acute{\mathbf{P}}$, $\grave{\mathbf{P}}$, $\mathbf{R}$ and $\mathbf{L}$ (for the definitions of these and other relevant operators on group classes we refer to the first chapter of [35]). It is easy to prove that any $\overline{\mathfrak{D}}$-group is locally graded, and that the class $\overline{\mathfrak{D}}$ is closed with respect to forming subgroups. Moreover, N.S. Černikov proved that every $\overline{\mathfrak{D}}$-group with finite rank contains a locally soluble subgroup of finite index (see [4]).

Finally, we need to notice that in [34] it is actually proved that, for an infinite locally finite field K, the groups $\mathrm{PSL}(2,\ K)$ and $\mathrm{Sz}(K)$ contain proper subgroups of infinite rank which do not have a normal hypercentral subgroup with Černikov quotient. Thus the following lemma is an easy consequence of this remark, Theorem A of [12] and the already quoted fact that periodic automorphism groups of Černikov groups are themselves Černikov groups.

Lemma 2. *Let $\mathfrak{X}$ be a class of groups satisfying* (c) *and let G be a locally (soluble-by-finite) group with all proper subgroups of infinite rank satisfying $\mathfrak{X}$. Then G contains a normal locally soluble subgroup of finite index.*

Theorem 7. *Let $\mathfrak{X}$ be a group class satisfying* (b),(c') *and let G be a $\overline{\mathfrak{D}}$-group of infinite rank whose proper subgroups of infinite rank have the property $\mathfrak{X}$. Then either G is soluble without proper subgroups of finite index, or G is not finitely generated and all proper subgroups of G are $\mathfrak{X}$-groups. In particular, if* (a) *holds, then G is actually an $\mathfrak{X}$-group.*

Proof. Let N be any proper normal subgroup of G having finite index. Then there are G-invariant subgroups $L \leq M$ of N such that L is Černikov, M/L is quasicentral in N/L and N/M has finite rank. An easy application of Lemma 6 of [8] shows the existence of a G-invariant subgroup S/L of M/L such that S/L and G/S have both infinite rank. Thus if E is any subgroup of finite rank, it follows that SE is a proper subgroup of G having infinite rank, and hence all proper subgroups of G have the $\mathfrak{X}$-property. Moreover, if G were finitely generated, then it would be easily seen to be of finite rank, being groups in $\mathfrak{X}$ locally of finite rank. Thus, if $\mathfrak{X}$ were local, the group G would belong to $\mathfrak{X}$.

Now we assume that G has no proper subgroup of finite index; in particular, G is not finitely generated and hence it is locally (soluble-by-finite) by [4]. Therefore, it follows from Lemma 2 that G contains a locally soluble normal subgroup of finite index and hence G is locally soluble.

Suppose that G has a proper normal subgroup N of infinite rank and let E be any subgroup of finite rank which is not an $\mathfrak{X}$-group. Then $NE = G$ and G/N has finite rank. Being locally soluble of finite rank it has a proper commutator subgroup (see Lemma 10.39 of [35]) so that $G' < G$ and G is easily seen to be soluble-by-finite and hence even soluble. Therefore we may assume further that every proper normal subgroup of G has finite rank.

Let Q be any normal subgroup of G, then it follows from Lemma 1 of [20] that Q is an $\mathfrak{X}$-group. Thus it contains a characteristic Černikov subgroup C such that Q/C has a normal hypercentral subgroup with Černikov quotient. Let D be any finite G-invariant subgroup of C. Then $G/C_G(D)$ is finite, so that $G = C_G(D)$. It is therefore safe to assume that in $G/Z(G)$ all proper normal subgroups have a normal hypercentral subgroup with Černikov quotient. Moreover, the Hirsch-Plotkin radical $H/Z(G)$ of $G/Z(G)$ is such that in G/H all normal subgroups are Černikov, and therefore central subgroups. Thus $G = H$ and G is locally nilpotent.

As G is locally soluble, all proper normal subgroups are actually soluble of finite rank. Since G is the union of its proper normal subgroups, it follows from Theorem 6.38 of [35] that G is hypercentral, so it has a proper commutator subgroup, which still means that G is soluble.

Finally, assume that $\mathfrak{X}$ is local also in this case and let F be any finitely generated subgroup of G. Then FG'/G' is a proper normal subgroup of the divisible group G/G' and hence $F \leq FG'$ satisfies $\mathfrak{X}$, which means that G does so. The statement is proved. $\boxed{QED}$

As locally graded k-(quasi)hamiltonian groups satisfy also (c'), the above result applies in particular to these group classes.

Corollary 14. *Let k be any positive integer and let G be a $\overline{\mathfrak{D}}$-group of infinite rank whose proper subgroups of infinite rank are k-(quasi)hamiltonian. Then G is k-(quasi)hamiltonian.*

Finally, we notice that it was proved in [2] that the class of finite-by-abelian groups is such that any locally graded group whose proper subgroups are finite-by-abelian is either finite-by-abelian or has finite rank. This remark make possible to prove the following corollary.

Corollary 15. *Let G be a $\overline{\mathfrak{D}}$-group of infinite rank whose proper subgroups of infinite rank are finite-by-abelian. Then G is finite-by-abelian.*

Proof. It follows from Theorem 7 and our previous remark that we may assume G is soluble with no non-trivial finite homomorphic images. Thus G/G' is divisible and it is the union of an ascending chain of normal subgroups, i.e.

$$G/G' = \bigcup_{n \in \mathbb{N}} E_n/G'$$

for normal subgroups E_n. Each E_n is clearly finite-by-abelian and hence E_n' is central in G. Thus $G/Z(G)$ is the union of an ascending chain of abelian subgroups, and is therefore abelian. Thus G is nilpotent.

Now, G/G' must have infinite rank and hence it contains a subgroup N/G' of infinite rank such that G/N has also infinite rank. It follows that all proper subgroups of G are contained in a proper subgroup of infinite rank and hence are finite-by-abelian. If G were not finite-by-abelian itself, it would give a contradiction to [2]. The proof is complete. $\boxed{QED}$

As for finite-by-quasihamiltonian groups, it is not clear whether they share the same behavior of finite-by-abelian groups or not, but our guess is that they do; for now we leave this as an open question.

References

[1] R. Baer: *Abzählbar erkennbare gruppentheoretische Eigenschaften*, Math. Z. 79 (1962), 344–363.

[2] B. Bruno, R.E. Phillips: *On minimal conditions related to Miller-Moreno type groups*, Rend. Sem. Mat. Univ. Padova, 69 (1983), 153–168.

[3] S.N. Černikov: *Infinite nonabelian groups in which all infinite nonabelian subgroups are invariant*, Ukrainian Math. J., 23 (1971/2), 498–517.

[4] N.S. Černikov: *A theorem on groups of finite special rank*, Ukrainian Math. J., 42 (1990), 855–861.

[5] M. De Falco, F. de Giovanni, H. Heineken, C. Musella: *Normality in uncountable groups*, Adv. Group Theory Appl., 3 (2017), 13–29.

[6] M. De Falco, F. de Giovanni, C. Musella: *Groups whose finite homomorphic images are metahamiltonian*, Comm. Algebra, 37 (2009), 2468–2476.

[7] M. De Falco, F. de Giovanni, C. Musella: *Metahamiltonian groups and related topics*, Int. J. Group Theory 2 (2013), 117–129.

[8] M. De Falco, F. de Giovanni, C. Musella: *Groups with normality conditions for subgroups of infinite rank*, Publ. Mat., 58 (2014), 331–340.

[9] M. De Falco, F. de Giovanni, C. Musella, R. Schmidt: *Groups in which every non-abelian subgroup is permutable*, Rend. Circ. Mat. Palermo (2), 52 (2003), 70–76.

[10] M. De Falco, F. de Giovanni, C. Musella, Y.P. Sysak: *Groups in which every subgroup is nearly permutable*, Forum Math., 15 (2003), 665–677.

[11] M.R. Dixon, M.J. Evans, H. Smith: *Groups with all proper subgroups nilpotent-by-finite rank*, Arch. Math. (Basel), 75 (2000), 81–91.

[12] M.R. Dixon, M.J. Evans, H. Smith: *Groups with all proper subgroups soluble-by-finite rank*, J. Algebra, 289 (2005), 135–147.

[13] M.R. Dixon, M. Ferrara, Z.Y. Karatas, M. Trombetti: *Groups with all subgroups permutable or soluble of finite rank*, J. Algebra, 549 (2020), 195–214.

[14] M.R. Dixon, M. Ferrara, M. Trombetti: *Groups in which all subgroups of infinite special rank have bounded near defect*, Comm. Algebra, 46 (2018), 5416–5426.

[15] D. Esposito, F. de Giovanni, M. Trombetti: *Groups whose non-normal subgroups are metahamiltonian*, Bull. Austral. Math. Soc., 102 (2020), 96–103.

[16] D. Esposito, F. de Giovanni, M. Trombetti: *Some trends in the theory of groups with finitely many normalizes*, Ricerche Mat., 69 (2020), 357–365.

[17] M. Ferrara, M. Trombetti: *A countable-type theorem for uncountable groups*, Adv. Group Theory Appl., 3 (2017), 97–114.

[18] M. Ferrara, M. Trombetti: *Groups whose non-permutable subgroups are metaquasi-hamiltonian*, J. Group Theory, 23 (2020), 513–529.

[19] S. Franciosi, F. de Giovanni, Ya.P. Sysak: *Group with many FC-subgroups*, J. Algebra, 218 (1999), 165–182.

[20] F. de Giovanni, M. Trombetti: *Groups of infinite rank with a locally finite term in the lower central series*, Beitr. Algebra Geom., 56 (2015), 735–741.

[21] F. de Giovanni, M. Trombetti: *Uncountable groups with restrictions on subgroups of large cardinality*, J. Algebra. 447 (2016), 383–396.

[22] F. de Giovanni, M. Trombetti: *Countable recognizability and nilpotency properties of groups*, Rend. Circ. Mat. Palermo (2), 66 (2017), 399–412.

[23] F. de Giovanni, M. Trombetti: *Nilpotency in uncountable groups*, J. Austral. Math. Soc., 103 (2017), 59–69.

[24] F. de Giovanni, M. Trombetti: *Countably recognizable classes of groups with restricted conjugacy classes*, Int. J. Group Theory, 7 (2018), 5–16.

[25] F. de Giovanni, M. Trombetti: *Large characteristic subgroups with modular subgroup lattice*, Arch. Math. (Basel), 111 (2018), 123–128.

[26] F. de Giovanni, M. Trombetti: *The class of minimax groups is countably recognizable*, Monatsh. Math., 185 (2018), 81–86.

[27] F. de Giovanni, M. Trombetti: *Groups with restrictions on proper uncountable subgroups*, Studia Sci. Math. Hungarica, 56 (2019), 154–165.

[28] F. de Giovanni, M. Trombetti: *Cohopfian groups and accessible group classes*, to appear.

[29] F. de Giovanni, M. Trombetti: *Groups whose proper subgroups are metahamiltonian-by-finite*, Rocky Mountain J. Math., 50 (2020), 153–162.

[30] P.B. Kleidman, R.A. Wilson: *A characterization of some locally finite simple groups of Lie type*, Arch. Math. (Basel), 48 (1987), 10–14.

[31] J.C. Lennox, J.S: Wilson: *A note on permutable subgroups*, Arch. Math. (Basel), 28 (1977), 113–116.

[32] P. Longobardi, M. Maj, H. Smith: *A note on locally graded groups*, Rend. Sem. Mat. Univ. Padova, 94 (1995), 275–277.

[33] A.Yu. Ol'shanskii: Geometry of defining relations in groups, Kluwer Academy Publishers, London 1991.

[34] J. Otal, J.M. Peña: *Infinite locally finite groups of type $PSL(2,K)$ or $Sz(K)$ are not minimal under certain conditions*, Publ. Mat., 32 (1988), 43–47.

[35] D.J.S. Robinson: Finiteness conditions and generalized soluble groups, Springer, Berlin 1972.

[36] R. Schmidt: Subgroup lattices of groups, de Gruyter, Berlin 1994.